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IMPACT AGAINST A SPINDLE-SHAPED SOLID BODY SUBMERGED IN A LIQUID OF INFINITE DEPTH

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The problem under consideration is that of vertical impact against a spindle-shaped solid body obtained by revolving a circular arc around the line passing through the extreme points of the arc, with half the body submerged in an ideal incompressible liquid of infinite depth. After making an odd extension of the potential through the free surface, the problem is split into questions of translation and rotation. The potential of liquid motion caused by the translational motion of the body is found in quadratures in bipolar coordinates. In the case of rotation, the problem is reduced to solving an integral Fredholm equation of the second order. The dependence of the apparent mass of the liquid and the apparent moment of inertia on the problem's parameters a and b (a is the radius of the arc of revolution and b is the coordinate of the arc's center) is studied. For a ball ($b = 0$), the result coincides with the classical result [1]. When $b = a$, agreement with the limiting case of a degenerate torus [2] is obtained. A condition for nonseparation of the impact is derived. The problem of flow around these surfaces was dealt with in [3].

1. Statement of the Problem. The potential Φ of velocities acquired by the liquid particles at the moment immediately following impact satisfies the Laplace equation in all the space occupied by the liquid [4]: $\Delta\Phi = 0$. It also satisfies the boundary conditions on the wet surface of the body and on the free surface of the liquid

$$\frac{\partial\Phi}{\partial n} = v_0 n_z + \omega(z n_x - x n_z), \quad \Phi = 0.$$

Here, v_0 and ω are translational and angular velocities; n_x and n_z are the projections of the outer normal vector on the x and z axes of the Cartesian coordinates. At infinity, $\Phi \to 0$. Making an odd extension of the potential Φ through the free surface, we come to the outer Neumann problem. The function Φ has the form

$$\Phi = v_0\Phi_1 + \omega\Phi_2, \tag{1.1}$$

where Φ_1 is the potential of liquid motion caused by the translational motion of the body along the symmetry axis z with unit velocity; Φ_2 corresponds to rotation of the body around the y axis with a constant angular velocity equal to unity. The apparent liquid mass m and moment J of inertia are determined from the formuls

$$m = -\rho \iint_S \Phi_1 \frac{\partial\Phi_1}{\partial n}\, ds, \quad J = -\rho \iint_S \Phi_2 \frac{\partial\Phi_2}{\partial n}\, ds \tag{1.2}$$

(S is the wet surface of the body and ρ is the liquid density).

2. Translational Motion. In the rotationally symmetric case, a stream function ψ can be introduced

$$\frac{\partial\Phi_1}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial z}, \quad \frac{\partial\Phi_1}{\partial z} = -\frac{1}{r}\frac{\partial\psi}{\partial r}.$$

To find ψ, we use bipolar coordinates α, β, and φ [5, p. 281] connected with the cylindrical coordinates r and z through the relations

$$r = \frac{c\sin\alpha}{\cosh\beta - \cos\alpha}, \quad z = \frac{c\sinh\beta}{\cosh\beta - \cos\alpha} \tag{2.1}$$

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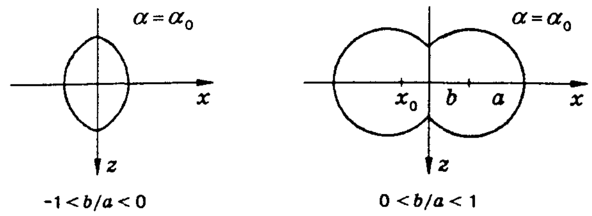

Fig. 1

(c is a scale factor, $0 \leqslant \alpha < \pi$, $-\infty < \beta < \infty$, $-\pi < \varphi \leqslant \pi$).
The coordinate system in question is suitable for solving boundary value problems of the potential theory for the region bounded by a spindle-shaped surface of revolution $\alpha = \alpha_0$ (Fig. 1) as well as for the region located between two nonintersecting spheres. The angular coordinate is denoted by φ. The constants c and α_0 can be expressed through the parameters a and b of the problem: $c = a \sin \alpha_0$ and $b = a \cos \alpha_0$. For the stream function $\psi = \psi(\alpha, \beta)$, we have the following boundary value problem in bipolar coordinates:

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{r}\frac{\partial \psi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{r}\frac{\partial \psi}{\partial \beta}\right) = 0, \quad \psi(\alpha_0, \beta) = -\frac{r^2}{2} = -\frac{c^2 \sin^2 \alpha_0}{2(\cosh \beta - \cos \alpha_0)^2}, \quad \psi|_\infty = 0. \qquad (2.2)$$

The exterior of the spindle-shaped surface corresponds with the region $0 \leqslant \alpha < \alpha_0$, $-\infty < \beta < \infty$, and $-\pi < \varphi \leqslant \pi$. Separation of variables for problem (2.2) leads to a solution of the form

$$\psi(\alpha, \beta) = \frac{\sin \alpha}{\sqrt{2\cosh \beta - 2\cos \alpha}} \int_0^\infty g(\tau) \cos \beta\tau \, P^1_{-1/2+i\tau}(\cos \alpha)\, d\tau$$

[$P^1_{-1/2+i\tau}(\cos \alpha)$ is the associated Legendre function with a complex subscript and $g(\tau)$ is an arbitrary function]. The condition at infinity, where $\alpha, \beta \to 0$, is satisfied since the introduced Legendre function vanishes at $\alpha = 0$ and the factor in front of the integral is a bounded function. Satisfaction of the boundary condition leads to decomposition of the given function in the cosine Fourier integral

$$\int_0^\infty g(\tau) \cos \beta\tau \, P^1_{-1/2+i\tau}(\cos \alpha_0)\, d\tau = -\frac{2c^2 \sin \alpha_0}{(2\cosh \beta - 2\cos \alpha_0)^{3/2}}.$$

From this we find the unknown function after performing the Fourier transform:

$$g(\tau) = -2c^2 P^1_{-1/2+i\tau}(-\cos \alpha_0)/(P^1_{-1/2+i\tau}(\cos \alpha_0) \cosh \pi\tau).$$

The integral representation and the differentiation formula for the Legendre function [5, p. 239; 6, p. 263] were used here:

$$P_{-1/2+i\tau}(-\cos \alpha_0) = \frac{2\cosh \pi\tau}{\pi} \int_0^\infty \frac{\cos \beta\tau \, d\beta}{\sqrt{2\cosh \beta - 2\cos \alpha_0}}; \qquad (2.3)$$

$$\frac{d}{d\alpha_0} P_{-1/2+i\tau}(-\cos \alpha_0) = -P^1_{-1/2+i\tau}(-\cos \alpha_0). \qquad (2.4)$$

We write the final expression for the stream function ψ:

$$\psi(\alpha, \beta) = -\frac{2c^2 \sin \alpha}{\sqrt{2\cosh \beta - 2\cos \alpha}} \int_0^\infty \frac{\cos \beta\tau}{\cosh \pi\tau} \frac{P^1_{-1/2+i\tau}(-\cos \alpha_0)}{P^1_{-1/2+i\tau}(\cos \alpha_0)} P^1_{-1/2+i\tau}(\cos \alpha)\, d\tau.$$

(An analogous expression was obtained in the problem of flow [3], where the apparent mass of the liquid is

TABLE 1

b/a	λ	ν	R
−0.7	0.022	0.003	0.30
−0.5	0.112	0.009	0.21
−0.3	0.331	0.009	0.13
−0.1	0.745	0.002	0.04
0.1	1.426	0.004	0.04
0.3	2.456	0.067	0.12
0.5	3.919	0.288	0.19
0.7	5.914	0.827	0.27
0.8	7.145	1.282	0.30
0.9	8.553	1.903	0.33

determined next.) Taking into account the conditions on the free surface, we find the potential uniquely:

$$\Phi_1(\alpha,\beta) = \int_0^\beta \frac{1}{r}\frac{\partial\psi}{\partial\alpha}\,d\beta. \tag{2.5}$$

The following differentiation formula is known [7]:

$$h(\alpha,\tau) = \frac{d}{d\alpha}P^1_{-1/2+i\tau}(\cos\alpha) = -\cot\alpha\, P^1_{-1/2+i\tau}(\cos\alpha) + (\tau^2+1/4)\,P_{-1/2+i\tau}(\cos\alpha). \tag{2.6}$$

Using this formula, we obtain the equality for the integrand of (2.5)

$$\frac{1}{r}\frac{\partial\psi}{\partial\alpha} = -c\left[-\frac{\sin\alpha}{\sqrt{2\cosh\beta-2\cos\alpha}}\int_0^\infty \frac{\cos\beta\tau}{\cosh\pi\tau}\frac{P^1_{-1/2+i\tau}(-\cos\alpha_0)}{P^1_{-1/2+i\tau}(\cos\alpha_0)}P^1_{-1/2+i\tau}(\cos\alpha)\,d\tau\right.$$

$$\left.+\sqrt{2\cosh\beta-2\cos\alpha}\int_0^\infty \frac{(\tau^2+1/4)}{\cosh\pi\tau}\cos\beta\tau\frac{P^1_{-1/2+i\tau}(-\cos\alpha_0)}{P^1_{-1/2+i\tau}(\cos\alpha_0)}P_{-1/2+i\tau}(\cos\alpha)\,d\tau\right]. \tag{2.7}$$

We find the apparent mass of the liquid. Passing to bipolar coordinates (2.1) in integral (1.2) for m, using formulas (2.3)–(2.5) and (2.7), and computing the integrals of the elementary functions, we obtain finally $m = \rho a^3\lambda$, where the function $\lambda = \lambda(b/a)$ has the form

$$\lambda = -\frac{\pi}{24}\left[15\sin\alpha_0 - 11\sin^3\alpha_0 + (\pi-\alpha_0)\cos\alpha_0\,(12+3\cos 2\alpha_0)\right]$$

$$+2\pi^2\sin^4\alpha_0\int_0^\infty \frac{(\tau^2+1/4)}{\cosh^2\pi\tau}\frac{[P^1_{-1/2+i\tau}(-\cos\alpha_0)]^2}{P^1_{-1/2+i\tau}(\cos\alpha_0)}P_{-1/2+i\tau}(\cos\alpha_0)\,d\tau, \qquad \cos\alpha_0 = b/a.$$

For calculating the Legendre functions, their relation to the hypergeometric series was used. The numerical results were compared with the tables in [8, 9].

The dependence of the function λ on the parameter b/a of the problem (Fig. 2) was studied. For $b = 0$, agreement with the classical result for a ball is demonstrated: $\lambda = \pi/3$ [1]. When $b = a$ the result agrees with the limiting case of a degenerate torus: $\lambda \approx 10.158$ [2]. Numerical values of λ are presented in Table 1.

3. Rotation of the Body. We consider the problem of rotation of a spindle-shaped surface in an ideal liquid. The Laplace equation and the boundary condition in bipolar coordinates have the form

$$\frac{\partial}{\partial\alpha}\left(r\frac{\partial\Phi_2}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(r\frac{\partial\Phi_2}{\partial\beta}\right)+\frac{c}{\sin\alpha\,(\cosh\beta-\cos\alpha)}\frac{\partial^2\Phi_2}{\partial\varphi^2}=0, \qquad \left.\frac{\partial\Phi_2}{\partial\alpha}\right|_{\alpha=\alpha_0} = \cos\varphi\,\frac{c^2\cos\alpha_0\sinh\beta}{(\cosh\beta-\cos\alpha_0)^2},$$

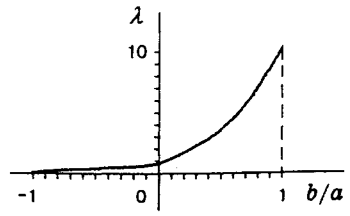

Fig. 2

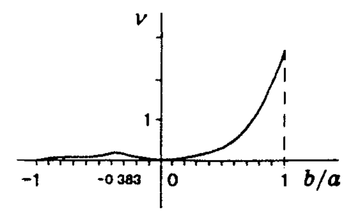

Fig. 3

where the following equality is used:

$$\frac{\partial \Phi_2}{\partial n} = -\frac{\cosh\beta - \cos\alpha_0}{c}\frac{\partial \Phi_2}{\partial \alpha}\bigg|_{\alpha=\alpha_0}.$$

Define the function Φ_2 as

$$\Phi_2(\alpha, \beta, \varphi) = \frac{4}{3}c^2 \cot\alpha_0 \cos\varphi\sqrt{2\cosh\beta - 2\cos\alpha}\int_0^\infty f(\tau)\sin\beta\tau P^1_{-1/2+i\tau}(\cos\alpha)\,d\tau. \tag{3.1}$$

Satisfying the boundary condition and taking into account (2.6), we get the equation

$$\frac{\sin\alpha_0}{2\cosh\beta - 2\cos\alpha_0}\int_0^\infty f(\tau)\sin\beta\tau P^1_{-1/2+i\tau}(\cos\alpha_0)\,d\tau + \int_0^\infty f(\tau)\sin\beta\tau\, h(\alpha_0, \tau)\,d\tau = \frac{3\sin\alpha_0\sinh\beta}{(2\cosh\beta - 2\cos\alpha_0)^{5/2}}.$$

Applying the Fourier transform to the last equation and using formulas (2.3) and (2.4) together with the value of the integral [10]

$$\int_0^\infty \frac{\cos z\beta\, d\beta}{\cosh\beta - \cos\alpha_0} = \frac{\pi}{\sin\alpha_0}\frac{\sinh(\pi - \alpha_0)z}{\sinh\pi z},$$

we arrive at a second-order integral Fredholm equation

$$y(\tau) + \int_0^\infty y(s)r(s)[k(\tau - s) - k(\tau + s)]\,ds = \frac{\tau P^1_{-1/2+i\tau}(-\cos\alpha_0)}{\cosh\pi\tau},$$
$$y(s) = f(s)h(\alpha_0, s), \quad r(s) = \frac{P^1_{-1/2+is}(\cos\alpha_0)}{2h(\alpha_0, s)}, \quad k(s) = \frac{\sinh(\pi - \alpha_0)s}{\sinh\pi s}. \tag{3.2}$$

We now calculate the apparent moment of inertia. Passing to bipolar coordinates in integral (1.2) for J and using formulas (2.3), (2.4), and (3.1), we obtain finally $J = \rho a^5\nu$, where the function $\nu = \nu(b/a)$ has the form

$$\nu = \frac{32\pi^2}{9}\sin^4\alpha_0\cos^2\alpha_0\int_0^\infty \frac{y(\tau)r(\tau)\tau P^1_{-1/2+i\tau}(-\cos\alpha_0)}{\cosh\pi\tau}\,d\tau$$

[$y(\tau)$ is the solution of integral equation (3.2)].

Figure 3 shows the dependence of the function ν on the characteristic parameter b/a of the problem. The function ν has two extremum points: a local minimum at $b = 0$ and a local maximum at $b/a \approx -0.383$. In the former case, the apparent moment of inertia is equal to zero, which is a well-known result [1]. The approximate value of the local maximum is 0.010. When $b = a$, the result agrees with the limiting case of a degenerate torus: $\nu \approx 2.728$. Numerical values of the function ν are presented in Table 1.

4. Condition of Nonseparation of the Impact. The coordinate of the point of application of the momentum is found from the formula

$$x_0 = -J\omega/mv_0. \tag{4.1}$$

The condition of nonseparation is that the momentum pressure $P_t = -\rho\Phi$ is nonnegative everywhere on the wet surface of the solid body. It seems obvious that separation starts in the vicinity of that point on the surface of the solid body which is most distant from the point of application of the momentum: $x = a + b$. This is confirmed by calculations for the surfaces $0 < b/a < 1$ (see Fig. 1).

Thus, for the surfaces mentioned, the condition of nonseparation is equivalent to nonnegativity of the momentum pressure in some vicinity of the point $x = a + b$. This, in turn, is equivalent to nonnegativity of the derivative of the momentum pressure with respect to the normal to the free surface at that point

$$\left.\frac{dP_t}{d\beta}\right|_{\substack{\beta=0\\ \varphi=0}} \geqslant 0. \tag{4.2}$$

Taking into account formulas (1.1), (2.5), (2.7), and (3.1) and calculating derivative (4.2), we obtain

$$v_0 A - \omega a \cos\alpha_0 B \geqslant 0,$$

$$A = -\frac{\cot^2(\alpha_0/2)}{8\sin(\alpha_0/2)} + \int_0^\infty \frac{(\tau^2+1/4)}{\cosh \pi\tau} \frac{P^1_{-1/2+i\tau}(-\cos\alpha_0)}{P^1_{-1/2+i\tau}(\cos\alpha_0)} P_{-1/2+i\tau}(\cos\alpha_0)\, d\tau, \tag{4.3}$$

$$B = \frac{4}{3}\int_0^\infty \tau f(\tau) P^1_{-1/2+i\tau}(\cos\alpha_0)\, d\tau.$$

For the surface $-1 < b/a < 0$ ($\pi/2 < \alpha_0 < \pi$) (see Fig. 1), the assumption about the separation point proves to be invalid. In this case, separation starts in the vicinity of that point on the surface of the solid body which is closest to the point of application of the momentum: $x = -a - b$. The condition of nonseparation in this case is determined from formula (4.3), where the absolute value of the function $\cos\alpha_0$ must be taken.

It was established numerically that the functions A and B are nonnegative. From (4.1) and (4.3), we get the necessary and sufficient condition of nonseparation of the impact in the form of estimate $|x_0| \leqslant aR$. The function $R = R(b/a)$ has the form

$$R = \frac{\nu}{\lambda} \frac{A}{|\cos\alpha_0| B}.$$

The numerical values of the function R are presented in the Table 1. When $a = b$, the result agrees with the limiting case of a degenerate torus: $R \approx 0.36$.

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